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The 'emergent scaling' phenomenon and the dielectric properties of random resistor–capacitor networks

R Bouamrane[1] **and D P Almond**[2,3]

[1] LEPM, Département de Physique, USTO-MB, BP 1505 El M'Naouer, Oran 31000, Algeria
[2] Department of Engineering and Applied Science, University of Bath, Bath BA2 7AY, UK

E-mail: mssdpa@bath.ac.uk

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Abstract
An efficient algorithm, based on the Frank–Lobb reduction scheme, for calculating the equivalent dielectric properties of very large random resistor–capacitor (R–C) networks has been developed. It has been used to investigate the network size and composition dependence of dielectric properties and their statistical variability. The dielectric properties of 256 samples of random networks containing: 512, 2048, 8192 and 32 768 components distributed randomly in the ratios 60% R–40% C, 50% R–50% C and 40% R–60% C have been computed. It has been found that these properties exhibit the anomalous power law dependences on frequency known as the 'universal dielectric response' (UDR). Attention is drawn to the contrast between frequency ranges across which percolation determines dielectric response, where considerable variability is found amongst the samples, and those across which power laws define response where very little variability is found between samples. It is concluded that the power law UDRs are emergent properties of large random R–C networks.

1. Introduction

One of the most interesting aspects of the electrical properties of solids is the anomalous power law dependence on frequency of dielectric properties and ac conductivity—this has been called the 'universal dielectric response' (UDR) [1, 2]. It has been shown that these features are reproduced in simulations of the electrical characteristics of large random networks of simple resistors and capacitors [3, 4]. This type of modelling has gained interest from experimental researcher involved in developing composites with required dielectric properties at radio and microwave frequencies [5, 6]. Such composites are typically a mixture of a non-conducting medium (wax or epoxy) and conducting particles (graphite or metallic particles). It is also, more

[3] Author to whom any correspondence should be addressed.

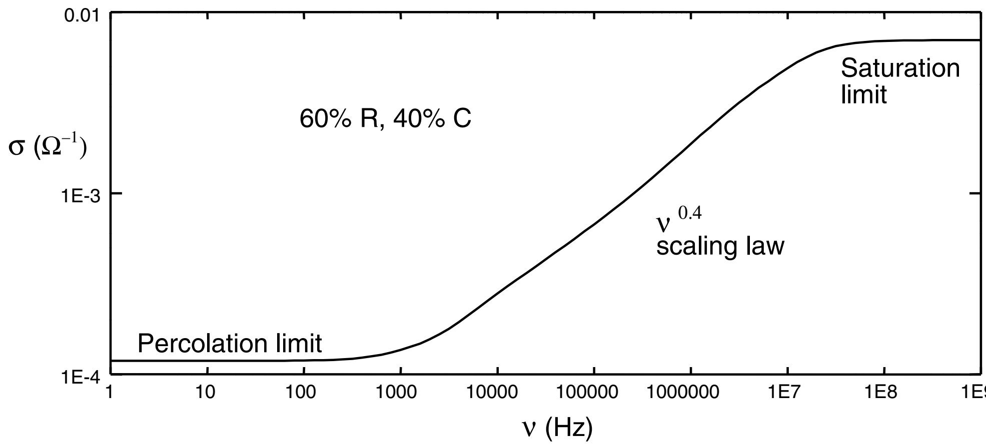

Figure 1. Ac conductivity for a 512 component network, 60% R–40% C [3].

generally, highly significant that complex structures of microscopically simple components exhibit the very widely observed UDR.

The simulations in the previous work [3, 4] were accomplished using commercial SPICE-based electrical circuit simulation software[4]. An example of the results obtained, included here for reference as figure 1, is the ac conductivity of a square network of 512 randomly positioned components, of which 60% were 1 kΩ resistors and 40% were 1 nF capacitors. The frequency dependence of the conductivity divides into three regions. At low frequencies, where the ac conductance of the capacitors, ωC, is much less than that of the resistors, the network conductivity is dominated by a random percolation path of resistors across the network. At intermediate frequencies, where the ac conductances of the capacitors and resistors become comparable, both component types contribute to network conductivity. In this frequency range, conductivity rises as a power of frequency, with the power law exponent equal to the capacitive component proportion—here 0.4. At high frequencies, the ac conductances of the capacitors far exceed those of the resistors and the network conductivity is again determined by the random distribution of all the resistors, effectively bound together by shorting capacitors. Note that the conductivity does not diverge at high frequencies because the proportion of capacitors is substantially less than the 50% necessary for percolation in a two-dimensional (2D) network. Simulations of capacitor-rich networks produced dielectric characteristics that exhibited the same three regions and, in particular, the anomalous power law dependences that have been called the 'UDR'.

In this paper, we present further numerical studies of the dielectric properties of 2D resistor–capacitor (R–C) networks making use of the highly efficient Franck–Lobb (FL) algorithm [7, 8]. This has enabled studies to be made of large numbers of networks to examine statistical variability and to significantly increase the sizes of the networks studied.

2. The model and the calculation

We consider a network with $L \times L$ nodes and therefore having $2 \times L \times L$ components. An example network with $L = 4$ and thus with 32 components is shown in figure 2. The dielectric response $\varepsilon(\omega)$ of this network is obtained through the equivalent complex impedance $Z_{eq}(\omega)$

[4] SIMetrix, Newbury Technology Ltd, Thatcham, Berks RG18 4LZ, UK.

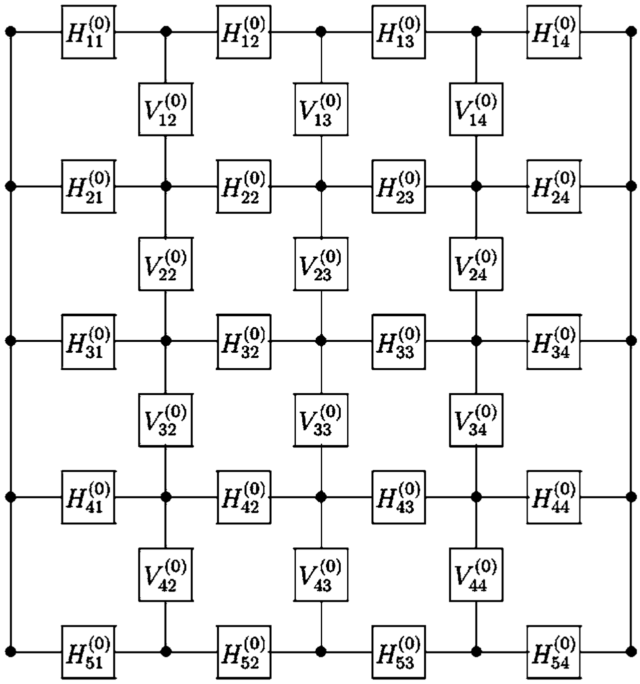

Figure 2. The initial 4×4 network.

calculated using the FL reduction scheme. The basic numerical technique associated with this algorithm involves a succession of triangle–star and star–triangle transformations. Figure 3 illustrates how one can remove the impedance $\boldsymbol{D}$ and move it across a node by first using a triangle–star transformation (move 1–2) and then a star–triangle transformation (move 3–4) and get the new impedance $\boldsymbol{D}'$.

In the following, the upper index n denotes the nth new impedance value taken by the bond after an FL transformation, with ($n = 0$) corresponding to the original bond impedance value which is either $Z_C = 1/(\mathrm{i}C\omega)$ or $Z_R = R$.

The FL reduction scheme starts with the triangle ($H_{11}^{(0)}$, $V_{12}^{(0)}$, $H_{21}^{(0)}$) which is transformed into a star ($\boldsymbol{Y}$, $V_{12}^{(1)}$, $H_{21}^{(1)}$) and then the star ($\boldsymbol{Y}$, $V_{22}^{(0)}$, $H_{22}^{(0)}$) back into a triangle ($\boldsymbol{D}$, $V_{22}^{(1)}$, $H_{22}^{(1)}$). This procedure is repeated in order to propagate the diagonal bond $\boldsymbol{D}$ down right (see figure 4). The last diagonal bond $\boldsymbol{D}$ is absorbed by combining it in parallel with $H_{54}^{(0)}$ which becomes $H_{54}^{(1)}$. The above sequence eliminates bond $H_{11}^{(0)}$ (represented by a dashed box in figure 4). A similar sequence starting with the triangle ($H_{21}^{(1)}$, $V_{22}^{(1)}$, $H_{31}^{(0)}$) removes the impedance $H_{21}^{(1)}$. Repetition of this procedure enables whole columns of impedances to be eliminated.

The equivalent impedance, $Z_{eq}(\omega)$, of the fully reduced network becomes:

$$Z_{eq}(\omega) = H_{51}^{(1)} + H_{52}^{(4)} + H_{53}^{(6)} + H_{54}^{(7)}$$

and the dielectric response is then obtained from the following relation:

$$\varepsilon(\omega) = \varepsilon'(\omega) + \mathrm{i}\varepsilon''(\omega) = 1/\mathrm{i}\varepsilon_0\omega Z_{eq}(\omega).$$

In the following section, the frequency (ν) is used instead of (ω). We have kept the parameters of [3] namely C = 1 nF and R = 1 kΩ. $\boldsymbol{D}$ and $\boldsymbol{Y}$ are temporary storage.

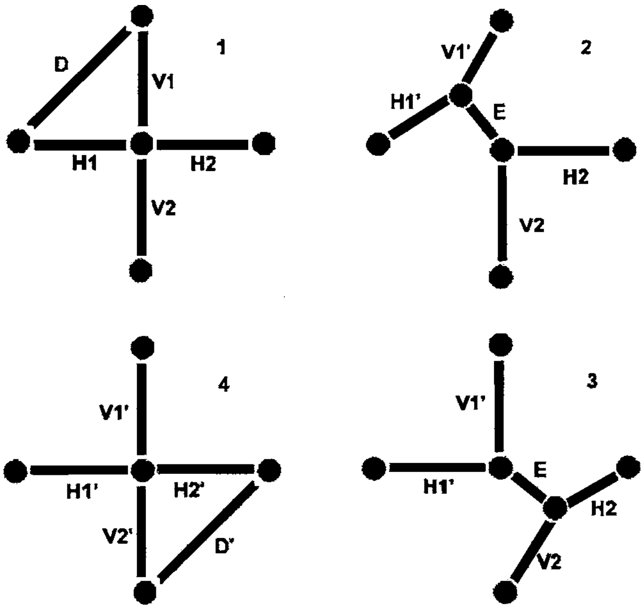

Figure 3. Basic transformation.

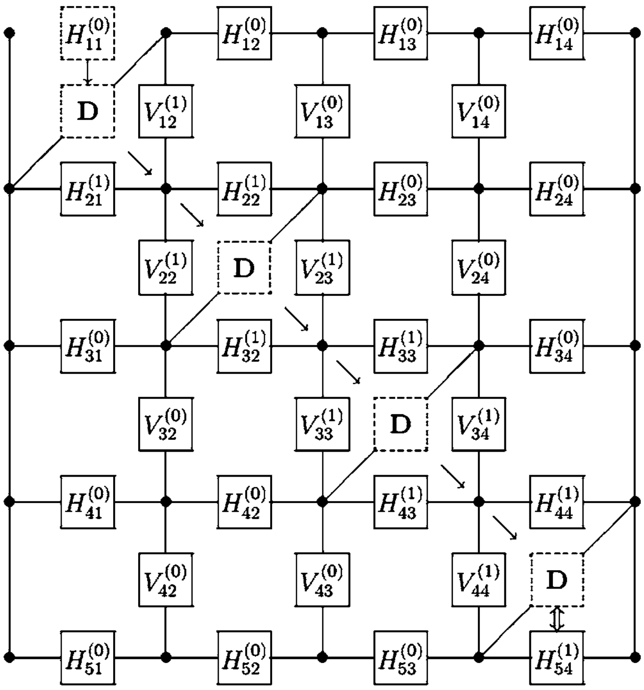

Figure 4. Reduction of the first cell.

3. Results and discussion

To examine the statistical variability of the results obtained in the earlier work [3, 4], the network responses of 256 separate networks of randomly positioned resistors and capacitors

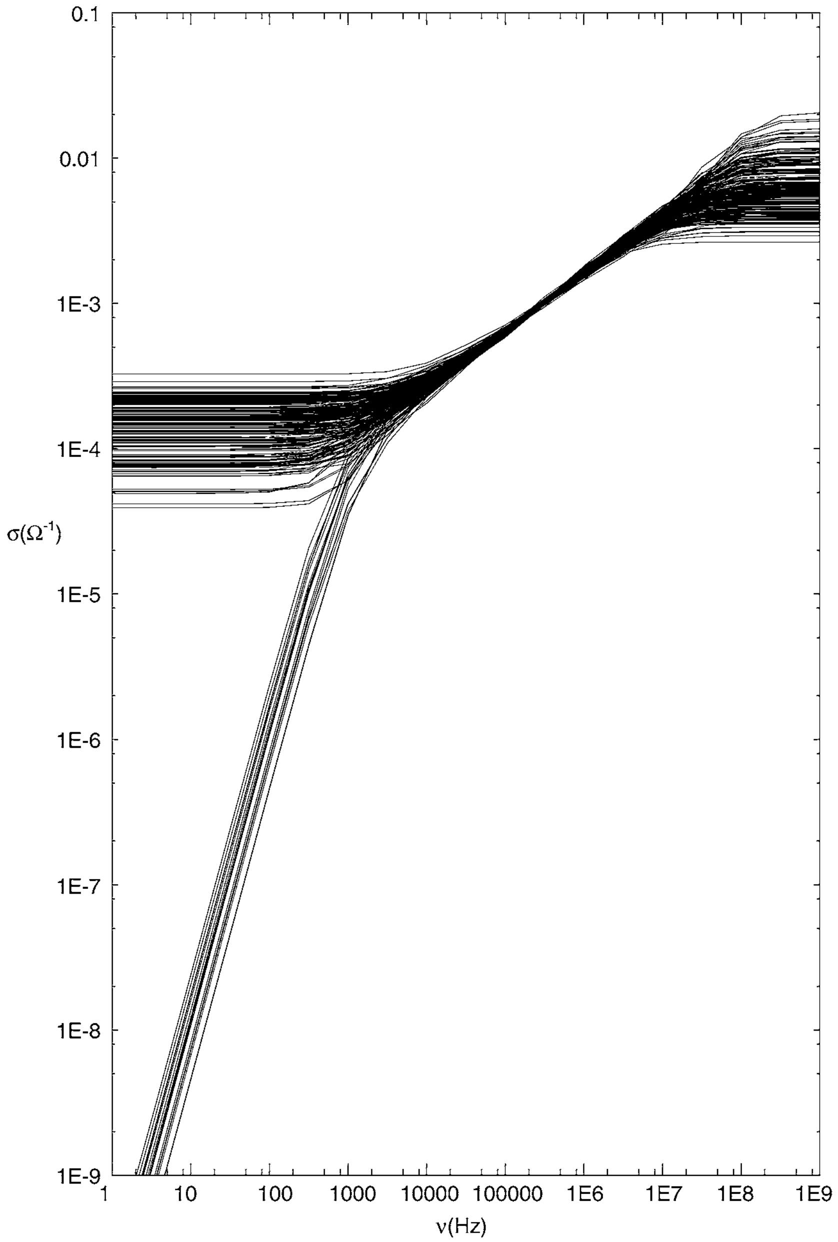

Figure 5. Ac conductivities of 256 R–C networks of 512 components (60% R, 40% C).

(512 components, 60% R and 40% C) were computed using the FL-based algorithm. The ac conductivities of all of these networks are shown in figure 5. These data are directly comparable with the data shown in figure 1. It is clear that ac conductivity exhibits broad distributions of magnitudes at high and low frequencies. This is expected as in these limits conductivity is determined by each specific arrangement of resistors that forms a percolation path across each random network. By contrast, at intermediate frequencies where conductivity rises as

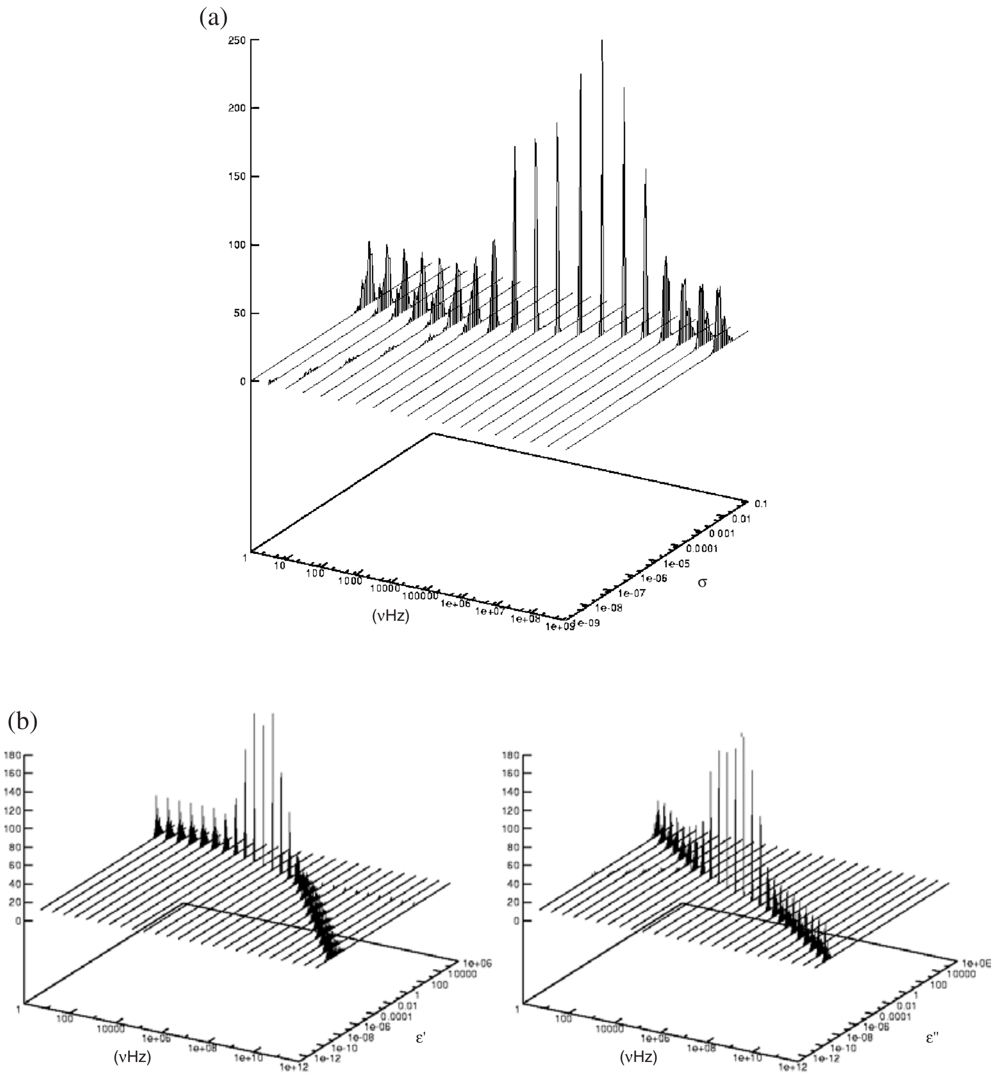

Figure 6. (a) Ac conductivity distributions versus frequency and (b) dielectric property distributions versus frequency of 256 R–C networks of 512 components (60% R, 40% C).

a power of frequency, there is very little spread in the conductivities computed for all 256 random networks. The results in figure 5 are re-plotted in figure 6(a) as the distribution of conductivity values obtained versus frequency. This representation emphasizes the difference between the broad distributions of values found at high and low frequency and the very narrow distributions obtained at intermediate frequencies. In figure 6(b), the corresponding $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$ are plotted.

Similar results have been obtained for dielectric networks containing 60% C–40% R and 50% R–50% C networks at the percolation limit. Again, very narrow distributions of $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$ are found across the 'UDR' power law region that was reproduced at intermediate frequencies.

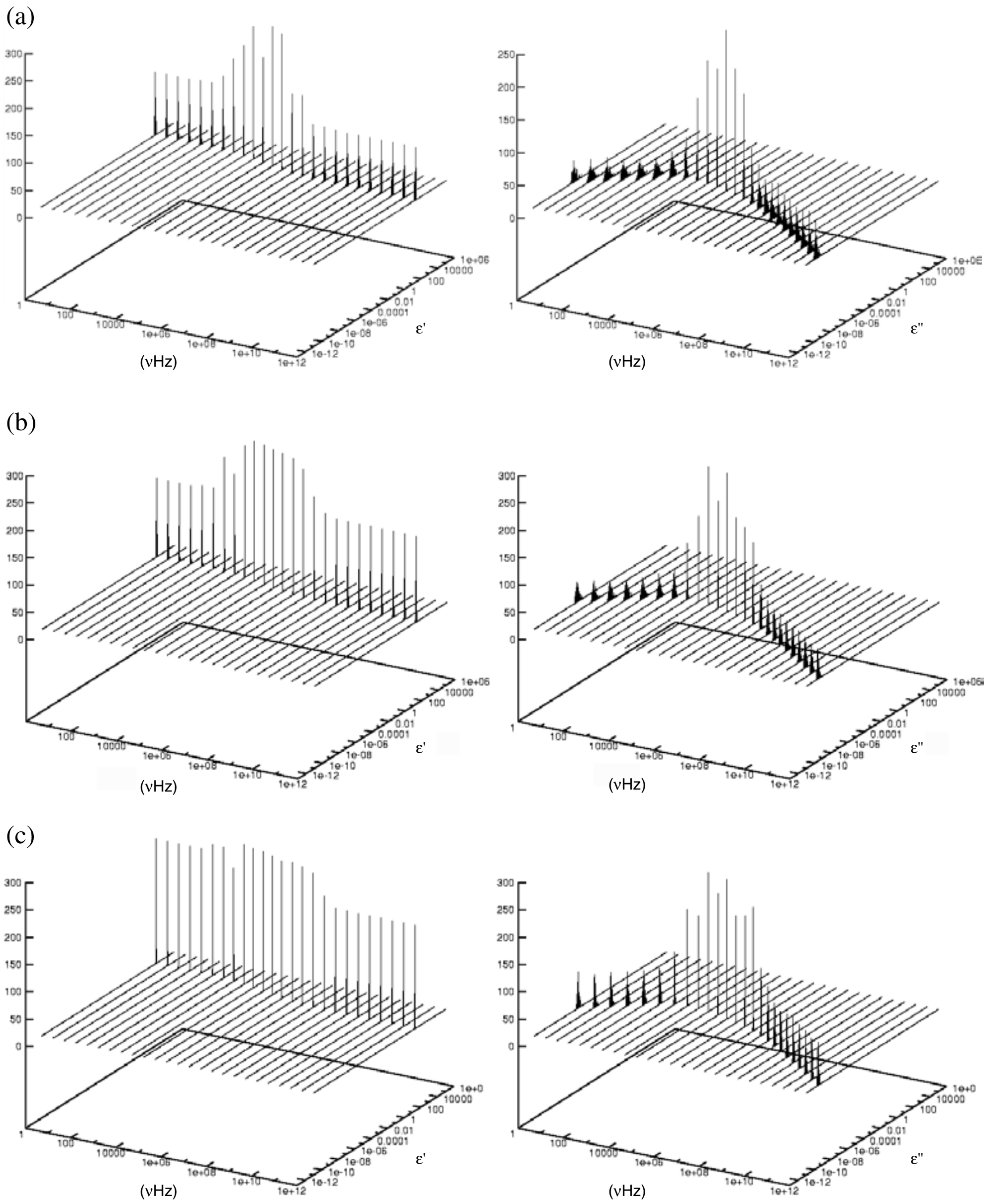

Figure 7. Dielectric property distributions versus frequency of 256 R–C networks of (a) 2048, (b) 8192 and (c) 32 768 components (60% R, 40% C).

The effect of network size has been examined by repeating the simulations for networks containing 2048, 8192 and 32 768 randomly positioned components. Again 256 independent random networks were generated and analysed using the algorithm. The general trend obtained is illustrated in figure 7 which shows the distributions of $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$ versus frequency obtained for 2048 (a), 8192 (b) and 32 768 (c) component dielectric networks of composition 60% C and 40% R. In all cases the intermediate frequency power law region is sharply defined. The scatter at low and high frequencies is reduced with increasing network size. Similar

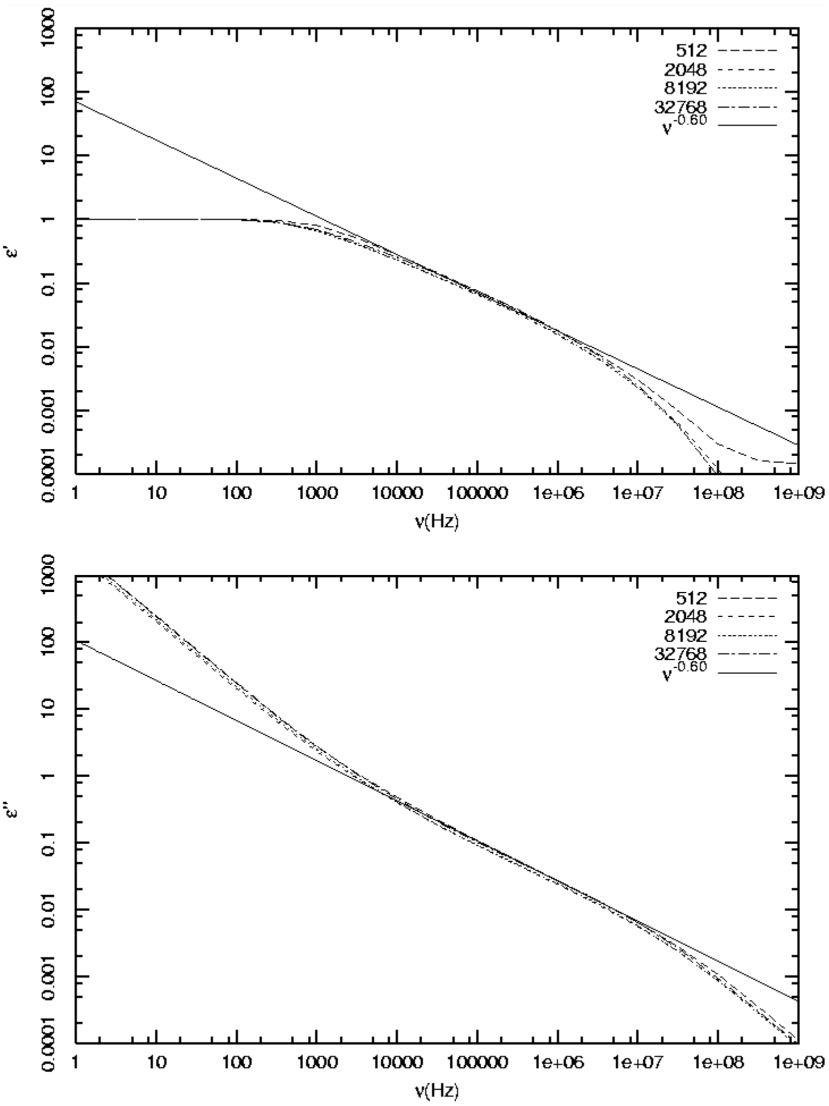

Figure 8. Average values of ε', normalized to the low frequency plateau and ε'' for 60% R–40% C component networks of sizes indicated in the figure.

behaviour was found for 60% R–40% C resistive networks but far less reduction in scatter was found for 50% R–50% C networks with compositions on the percolation limit.

In figures 8–10, the averages of the normalized dielectric constants ε' and ε'' versus ν are plotted. A rough fit seems to indicate a ν^{-c} dependence, where c is the resistor concentration, as found and explained in [3]. In figures 11 and 12, the averages of the real conductivity σ versus ν are shown for the 60% R–40% C and 50% R–50% C networks.

Simulation timings obtained using the FL algorithm and the SPICE-based software SIMetrix (see footnote 4) are shown for comparison in table 1.

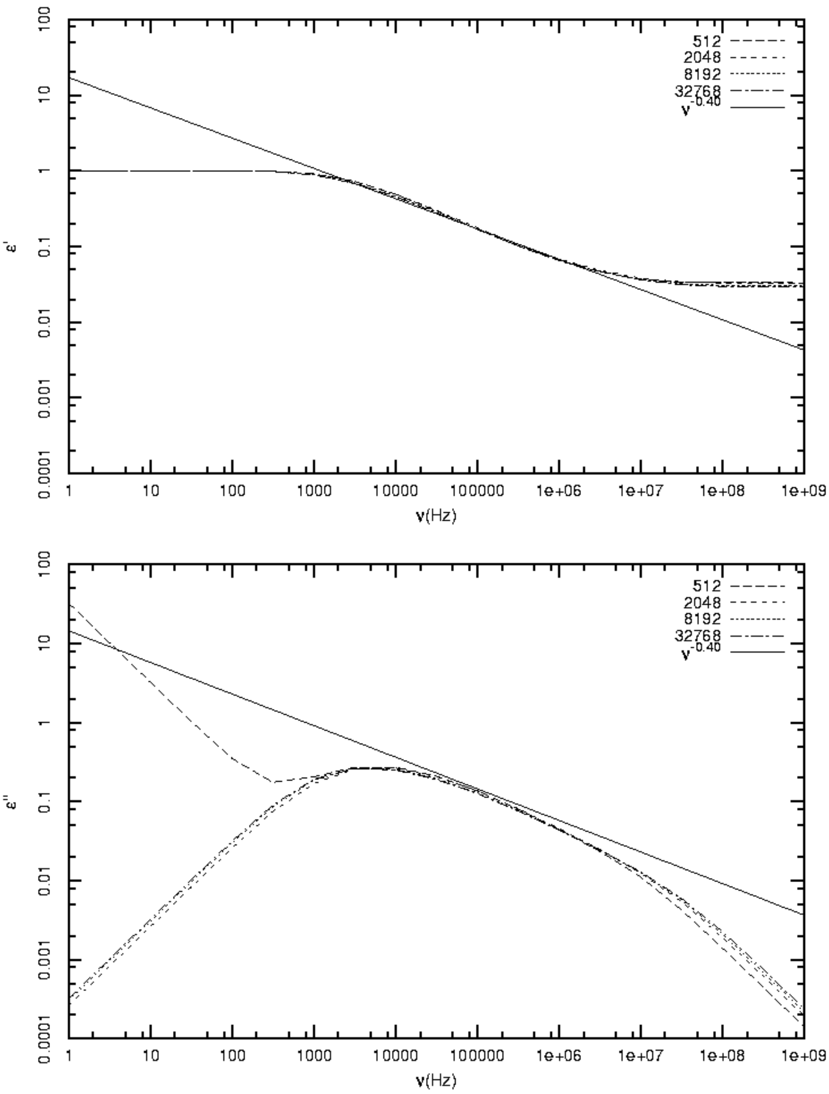

Figure 9. Average values of ε', normalized to the low frequency plateau and ε'' for 60% C–40% R component networks of sizes indicated in the figure.

Table 1. CPU time comparison for one sample network in the range 1 Hz–1 GHz with 10 frequency points computed per decade.

Size	This algorithm PIII 500 MHz (s)	SIMetrix (see footnote 4) PIII 500 MHz (s)
512	0.2	2.0
2 048	1.5	19.4
8 192	12.5	242.0
32 768	106.5	2547.0

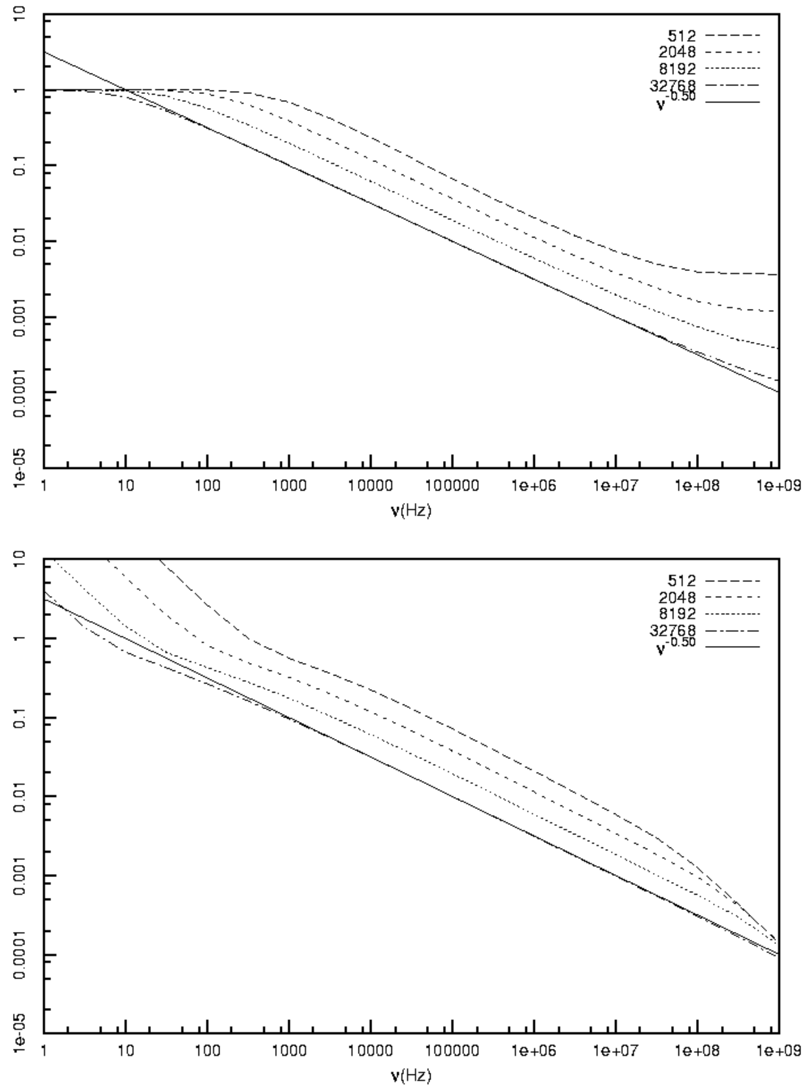

Figure 10. Average values of ε', normalized to the low frequency plateau and ε'' for 50% R–50% C component networks of sizes indicated in the figure.

4. Conclusions

In this work an efficient new algorithm for analysing the ac responses of very large networks of electrical components has been developed. It has been used to study the statistical variability and network size dependences of random networks of resistors and capacitors. The results confirm the contention [3, 4] that the anomalous power law dependences of dielectric permittivity and ac conductivity on frequency are intrinsic properties of random networks of resistors and capacitors. It has been found that frequency ranges across which these power law regions occur expand with network size. It has also been found that, for all network sizes, the electrical properties across this power law range are sharply defined in magnitude and character, showing little statistical variability and in contrast to the properties found at low and

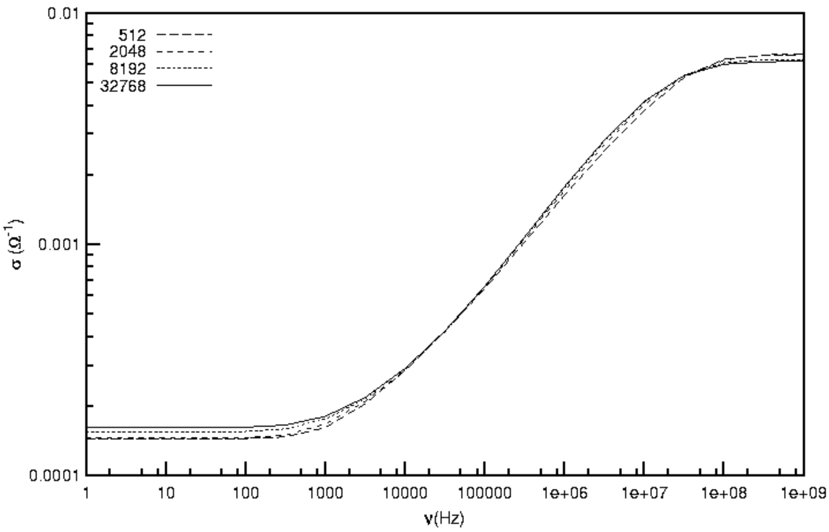

Figure 11. Averaged real conductivity $\sigma(\nu)$: 60% R–40% C component networks.

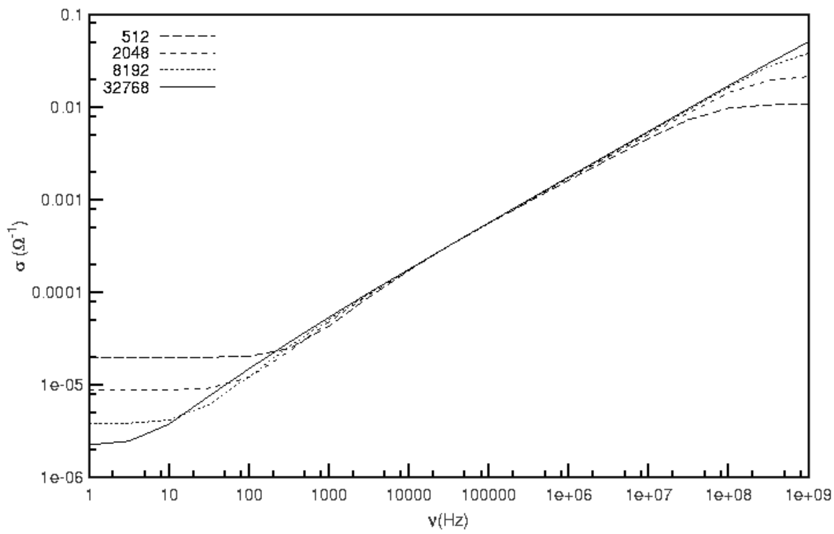

Figure 12. Averaged real conductivity $\sigma(\nu)$: 50% R–50% C component networks.

high frequencies. For these reasons it is concluded that the power law characteristics are an emergent property of random networks of resistive and capacitive components that we propose should be called ‘emergent scaling’.

The discovery of power law frequency dependence in the electrical responses of random networks provides a possible explanation for the UDR. For many of the materials that are found to exhibit UDR, it is plausible to suggest the presence of inhomogeneous microstructures that effectively form a random network of resistive and capacitive microstructural regions. The attraction of this explanation is that it does not necessitate the introduction of any ‘new physics’—UDR is an emergent property of a random network of conventional capacitive and resistive components. It remains to be seen whether the required resistive and capacitive regions can be identified in materials that exhibit UDR. It also remains to be seen whether the emergent scaling that has been found in two-dimensional networks is equally prominent in the characteristics of three-dimensional networks.

Emergent scaling may be of practical value in the design of composite electrical materials with required conductivities or dielectric properties for use across a particular frequency range. It provides a way of avoiding the intrinsic variability of percolation paths and associated problems of achieving manufacturing reproducibility. It may also reduce instances of component failure by electrical breakdown because in the emergent scaling region the whole network, in contrast to the tenuous routes employed in the percolation region, carries ac currents.

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